Meromorphic Extension of the Spherical Functions on a Class of Ordered Symmetric Spaces

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We discuss a conjecture of G. Ólafsson and A. Pasquale published in [14]. This conjecture gives the Bernstein-Sato polynomial associated with the Poisson kernel of the ordered (or non-compactly causal) symmetric spaces. The Bernstein-Sato polynomials allow to locate the singularities of the spherical functions on the considered spaces. We prove that this conjecture does not hold in general, and propose a slight improvement of it. Finally, we prove that the new conjecture holds for a class of ordered symmetric spaces, called both the Makarevič spaces of type I, and the satellite cones.

Key Words: spherical function; Bernstein-Sato polynomial; Jordan algebra; non-compactly causal symmetric space; symmetric cone.

1. INTRODUCTION

An ordered (or non-compactly causal) symmetric space G/H is a particular non-Riemannian symmetric space. Its specificity is to carry a partial order, which is invariant under the action of the Lie group G. This structure allows to develop a theory of spherical functions close to the theory on the Riemannian symmetric spaces (cf. [5]).

Roughly speaking, the spherical functions φ_λ play the role of the exponential functions in the Euclidean theory of the Fourier transform. They are labeled by the elements $\lambda \in \mathfrak{a}^*$ belonging to (the dual of) a Cartan subspace of the Lie algebra associated to G. There are two main differences between the spherical functions on the ordered symmetric space G/H and the ones on the Riemannian symmetric space G/K, essentialy due to the lack of compacity of the group H.

In one side, the φ_λ are defined on the futur of $1H$, $\{x \in G/H \mid x > 1H\}$, but are defined everywhere on G/K. In other side, the dependence of φ_λ on λ is meromorphic in the ordered case, instead of holomorphic in the Riemannian case. A natural problem, crucial to understand the

Laplace transform on ordered symmetric spaces, is therefore to locate the λ-singularties of φ_λ.

A strategy to get an explicit solution to problem has been proposed in [14]. This approach is based on an integral formula of φ_λ, valid for λ in a domain $\mathcal{E}$. This formula reduce the problem of meromorphic extension to the construction of Bernstein identities on a family of polynomials (cf. [3]). The authors of [14] have stated a conjecture (cf. conjecture 1) on the shape of these Bernstein identities.

In our article, we will prove (an improvement of) this conjecture on a family of ordered symmetric spaces, the satellite cones. The satellite cones are closely related to simple Euclidean Jordan algebras (cf. [6]). The proof of this conjecture will be based on the previous supplement of structure, in particular a result of [2] on Bernstein identities of the power function (theorem 4), and will also use the theory of the radial part of differential operators.

The section 2 is devoted to the presentation of the Jordan algebras, and the setting of the main notations. We introduce and classify the satellite cones in section 3 and we recall the properties of the spherical functions in section 4. In section 5, we provide a counter example to the conjecture of [14] (lemma 2), and propose a new conjecture (conjecture 3). The new conjecture seems weaker than the intial one, but is sufficent to study the singularties of φ_λ. Finally, we prove the conjecture 3 for the satellite cones in section 6, theorem 6, and deduce a set containing the polar set of φ_λ (theorem 7).

2. PRELIMINARIES ON JORDAN ALGEBRAS

Let V be a real, finite dimensional, *Euclidean*, and *simple Jordan algebra.* It means that V is a real vector space of finite dimension n, equipped with a commutative multiplication which satisfies the axiom :

$$x(x^2y) = x^2(xy), \quad \forall x, y \in V.$$

Moreover, V has no non-trivial ideal and is endowed with an associative positive-definite bilinear form.

There are three crucial applications associated to the algebra V : the *regular representation* L, the *quadratic representation* P, and the *Jordan triple system* $\{.,.,.\}$. They are defined by the relations :

$$\begin{aligned} L(x)y &= xy, \quad \forall x, y \in V, \\ P(x) &= 2L(x)^2 - L(x^2), \quad \forall x \in V, \\ \{x, y, z\} &= x(yz) - y(xz) + (xy)z, \quad \forall x, y, z \in V. \end{aligned}$$

The algebra V has a neutral element denoted by e. Let x be an element of V and consider the subalgebra $\mathbb{R}[x]$ of V, generated by e and the various powers of x. The algebra $\mathbb{R}[x]$ is associative, the dimension of $\mathbb{R}[x]$ defines the *rank* of x, and the *rank* r of V is definied by the maximum of the ranks of the elements of V. The trace $\mathrm{tr}(x)$ and the determinant $\Delta(x)$ are given by the trace and the determinant of the restriction of $L(x)$ to the space $\mathbb{R}[x]$. The *Peirce constant* d is an integer defined by the relation :

$$n = r + r(r-1)\frac{d}{2}.$$

If the determinant of x is non-zero, then x has an inverse in the associative algebra $\mathbb{R}[x]$, which defines the *inverse* of x in V.

There exists on V a bilinear, associative, positive-definite form, unique up to a positive scalar. We consider the following one :

$$\eta(x, y) := \mathrm{tr}(xy), \quad \forall x, y \in V.$$

Let $g \in \mathfrak{gl}(V)$ be an endomorphism. We will denote by g' the adjoint of g with respect to the form η.

Define the *structure group* $\mathrm{Str}(V)$ of the Jordan algebra V by :

$$\mathrm{Str}(V) := \{g \in \mathrm{GL}(V) : \forall x \in V, \quad gP(x)g' = P(g \cdot x)\}.$$

The group $\mathrm{Str}(V)$ is a closed subgroup of the linear group $\mathrm{GL}(V)$, and it is a reductive Lie group. The *group of automorphism* of V is a maximal compact subgroup of $\mathrm{Str}(V)$:

$$\mathrm{Aut}(V) = \{k \in \mathrm{Str}(V) \mid \forall x, y \in V, \quad k \cdot (xy) = (k \cdot x)(k \cdot y)\}.$$

The connected component of a subgoup H of $\mathrm{Str}(V)$ containing the neutral element e is denoted by H_o. In particular, we consider $G = \mathrm{Str}(V)_o$, and $K = \mathrm{Aut}(V)_o$. The Lie algebras associated with these two groups are respectively $\mathfrak{g}$ and $\mathfrak{k}$. They are subalgebras of the Lie algebra $\mathfrak{gl}(V)$. The involution θ of $\mathfrak{gl}(V)$ defined by $\theta(X) = -X'$ is a Cartan involution on $\mathfrak{g}$. The algebra of θ-fixed points is $\mathfrak{k}$, and $\mathfrak{p} = L(V)$ is the eigenspace of θ corresponding to the eigenvalue -1.

The eigenspaces of $L(x)$ are denoted by

$$V(x, \gamma) = \{y \in V \mid xy = \gamma y\}.$$

Let c be an idempotent of V (c satisfy $c^2 = c$). The following decomposition, orthogonal with respect to η, is the *Peirce decomposition* associated to c :

$$V = V(c, 1) \oplus V(c, 1/2) \oplus V(c, 0).$$

We define the *Frobenius operator* related to the idempotent c and the element $z \in V(c, 1/2)$:

$$\begin{aligned}
\mathfrak{F}(c, z) &= \exp(L(z) + 2[L(z), L(c)]) \\
&= \exp(2\{c, z, .\}) \\
&= I + 2\{c, z, .\} + 2\{c, z, .\}^2.
\end{aligned}$$

Let $x = x_0 + x_{1/2} + x_1$ be the Peirce decomposition of x. The Peirce decomposition of $y = \mathfrak{F}(c, z)(x)$ is given by :

$$\begin{aligned} y_1 &= x_1 \\ y_{1/2} &= 2zx_1 + x_{1/2} \\ y_0 &= 2(e-c)(z(zx_1)) + 2(e-c)(zx_{1/2}) + x_0. \end{aligned} \tag{1}$$

These results are described in [6], pages 106 and 107.

We choose a Jordan frame on V, denoted by $(c_i)_{i=1}^r$ (see [6], §IV.2.). The system $(c_i)_{i=1}^r$ contains r orthogonal idempotents :

$$\begin{aligned} c_i c_j &= \delta_{ij} c_i, \\ e &= c_1 + ... + c_r, \end{aligned}$$

where δ is the Kronecker symbol.

According to proposition IV.1.1 of [6], the following spaces are two simple Jordan subalgebras of V :

$$V_i = V(c_1 + ... + c_i, 1) \quad ; \quad V_{r-i}^* = V(c_1 + ... + c_i, 0), \quad i \in \{0, ..., r\}.$$

The orthogonal projection of V onto V_i (resp. V_i^*) is denoted by π_i (resp. π_i^*). The neutral element of V_i is $\pi_i e = e_i$ and the one of V_i^* is $\pi_i^* e = e_i^*$. We set $V^{i,r-i} = V(e_p, 1/2) = V(e_{r-i}^*, 1/2)$.

The composition of the π_i with the determinant of V_i is denoted by Δ_i. The Δ_i are the *principal minors* of the algebra V.

The vector space $\mathfrak{a} = L(\mathbb{R}c_1 + ... + \mathbb{R}c_r)$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$.

We identify the linear forms $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}_{\mathbb{C}}$ with $\mathbb{C}^r$, using $\mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}^r, \lambda \mapsto \tilde{\lambda}$ determined by the requierement :

$$\lambda(a_1 L(c_1) + ... + a_r L(c_r)) = \sum_{i=1}^{r} \vec{\lambda}_i a_i.$$

Let $\vec{\imath_i} = (\delta_{ij})_{j=1}^r \in \mathbb{C}^r$. Set

$$\alpha_{ij} := \left(\frac{1}{2}(\imath_j - \imath_i) \right),$$

and,

$$\mathfrak{g}_{\alpha_{ij}} = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a}, [H, X] = \alpha_{ij}(H)\}.$$

We have $\theta(\mathfrak{g}_{ij}) = \mathfrak{g}_{ji}$, (see [6], proposition VI.3.3), and

$$\mathfrak{g}_{ij} = \{\{c_i, z, .\} \mid z \in V_{ij} := V(c_i, 1/2) \cap V(c_j, 1/2)\}.$$

In particular, we notice that $\dim \mathfrak{g}_{ij} = d$. We can also see that the algebra $\mathfrak{n} = \oplus_{i<j} \mathfrak{g}_{ij}$ is the nilpotent algebra related to the choice of the positive Weyl chamber :

$$\mathfrak{a}^+ = \{L(a_1 c_1 + ... + a_r c_r) \mid a_1 \leq ... \leq a_r\}.$$

The restricted root system of $\mathfrak{g}$ with respect to the Cartan subspace $\mathfrak{a}$ is therefore $\Delta = \{\alpha_{ij} \mid i \neq j\}$, and the multiplicity of each root is d. The positive root system corresponding to the choice of $\mathfrak{a}^+$ is $\Delta^+ = \{\alpha_{ij} :\mid i < j\}$, and the corresponding simple root system is $\Pi = \{\alpha_i := \alpha_{ii+1} \mid i = 1, ..., r-1\}$. We consider the permutation group $\mathcal{W}$ of the set $\{1, ..., r\}$. The action of $w \in \mathcal{W}$ on $\mathfrak{a}^*$ is given by

$$w \cdot \alpha := (w \cdot \alpha) = (\alpha_{(w \cdot 1)} \mathbf{1}_1 + ... + \alpha_{(w \cdot r)} \mathbf{1}_r).$$

The group $\mathcal{W}$ is actually the Weyl group associated to Δ.

In brief, the restricted root system Δ is of type A_r, with multiplicity d.

We also consider the algebra $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$. Let A, N and $\overline{N}$ be the analytic subgroups of G respectively corresponding to $\mathfrak{a}$, $\mathfrak{n}$, and $\overline{\mathfrak{n}}$. The centralizer of $\mathfrak{a}$ in K is denoted by M.

We complete the notations. Define, for $i, j \in \{1, ..., r\}$ and $i \neq j$, the covector $L(h_{ij})$ of α_{ij} by

$$h_{ij} = 2\left(c_j - c_i\right).$$

Thus $\alpha_{ij}(L(h_{ij})) = 2$, and for all $L(h) \in \mathfrak{a}$,

$$\eta(h_{ij}, h) = 4\alpha_{ij}(L(h)).$$

We provide $\mathfrak{a}^*$ with the inner product $(\alpha_{ij}, \alpha_{kl}) = \eta(h_{ij}, h_{kl})$.

Finally, we construct the fundamental weights $\mu_i \in \mathfrak{a}^*$. Define $\vec{\mu}_i \in \mathbb{C}^r$ by $\vec{\mu}_{ij} = -1$ if $j \in \{1, ..., i\}$, and by 0 if $j \in \{i+1, ..., r\}$. The linear form μ_i actually satisfy :

$$(\mu_i, \alpha_j) = \delta_{ij}(\alpha_j, \alpha_j). \tag{2}$$

We fix two non-negative integers p and q such that $p + q = r$, and let $e' := e_p - e_q^*$. Notice that $e'e' = e$ and that $P(e')$ is an involution of K. On V, the 1-eigenspace of $P(e')$ is $V_p \oplus V_q^*$, and the -1-eigenspace is $V^{pq} = V(e_p, 1/2)$.

The involution of $GL(V)$ defined by $\tau(g) = P(e')\theta(g)P(e')$ stabilizes G. The derivative involution of τ is still denoted by τ. The 1-eigenspace (resp. -1-eigensapce) of τ on $\mathfrak{g}$ is $\mathfrak{h}$ (resp. $\mathfrak{q}$). The analytic subgoup of G corresponding to the Lie algebra $\mathfrak{h}$ is H.

We observe that θ and τ commute. Thus :

$$\mathfrak{g} = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{h} \cap \mathfrak{p} \oplus \mathfrak{q} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p}.$$

3. STRUCTURE OF THE SATELLITE CONES

The satellite cones correspond up to a Cayley transform to the category of Makarevič spaces of type I, defined by W. Bertram ([4], chapter XI). The term of satellite cone was introduced by Loos ([11]).

DEFINITION 1. The *satellite cones* are the various open G-orbits in V. Thus, these cones are on one side $\Omega' = G \cdot e'$, for $0 < p < r$ and $q = r - p$, and on the other side the two convex cones $\pm\Omega$, with $\Omega = G \cdot e$.

The classification of satellite cones is an easy consequence of the classification of simple real Euclidiean Jordan algebras (cf . [6], chapter V). Notice that the parameters (p, q, d) characterize a satellite cone :

V	rank	d	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$
$Sym(r,\mathbb{R})$	r	1	$\mathfrak{sl}(r,\mathbb{R}) \oplus \mathbb{R}$	$\mathfrak{so}(p,q)$	$\mathfrak{so}(r)$
$Herm(r,\mathbb{C})$	r	2	$\mathfrak{sl}(r,\mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}(p,q)$	$\mathfrak{su}(r)$
$Herm(r,\mathbb{H})$	r	4	$\mathfrak{su}^*(2r) \oplus \mathbb{R}$	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(r)$
$\mathbb{R} \times \mathbb{R}^{n-1}$	2	n-2	$\mathfrak{o}(n-1,1) \oplus \mathbb{R}$	$\mathfrak{o}(n-2,1)$	$\mathfrak{o}(n-1)$
$Herm(r,\mathbb{O})$	3	8	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{f}_4$

We are going to describe the ordered symmetric space structure of Ω'. A general reference on this theory is [5].

LEMMA 1. *The satellite cone Ω' is a non-compact symmetric space. It is provided with a structure of ordered symmetric space (also called non-compactly causal symmetric space). It is equivalent to the existence of a non-zero and non-central element in $(\mathfrak{p} \cap \mathfrak{q})^{H \cap K}$.*

Proof. The Lie group G is reductive and non-compact, endowed with an involutive non-trivial automorphism τ, which satisfy $G_0^\tau \subset H \subset G^\tau$. Hence, $G/H \approx \Omega'$ is a non-compact symmetric space.

Using the fact that $P(e')$ is involutive, we get $\forall x \in V$, $\forall t \in \mathbb{R}$,

$$\tau(P(\exp tx)) = P(P(e')\exp(tx)) = P(\exp(tP(e')x)). \tag{3}$$

The derivative of relation (3) is : $\tau(L(x)) = L(P(e')x)$. Thus, $L(x) \in \mathfrak{p} \cap \mathfrak{q}$ is equivalent to $P(e')x = x$, therefore $x \in V_p + V_q^*$. We obtain :

$$L(e') \in \mathfrak{p} \cap \mathfrak{q} = L(V_p + V_q^*).$$

Finally, for $h \in H \cap K$, and $x \in V$,

$$(\mathrm{Ad}(h)L(e'))x = h(e'(h^{-1}x)) = (he')x = e'x = L(e')x,$$

where the second equality comes from $h \in K$ and the third from $h \in H$. In conclusion, $L(e') \in (\mathfrak{p} \cap \mathfrak{q})^{H \cap K}$. ∎

We have in particular noticed that $L(e') \in \mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$.

Define :

$$\Delta_\epsilon = \{\alpha \in \Delta \mid \alpha(L(e')) = \epsilon\}, \quad \forall \epsilon \in \{1, 0, -1\}.$$

The set Δ_0 denotes the set of *compact roots*, and $\Delta_1 \cup \Delta_{-1}$ the set of *non-compact roots*. We observe that $\Delta_1 = \{\alpha_{ij} \mid i \leq p < j\} \subset \Delta^+$ and

$\Delta_{-1} \subset -\Delta^+$. We naturally define N_0, $\overline{N}_0$, N_1 and N_{-1} as the analytic nilpotent subgroups of G respectively associated to the sets of roots $\Delta_0 \cap \Delta^+$, $\Delta_0 \cap (-\Delta^+)$, Δ_1, Δ_{-1}. The *maximal cone* and the *minimal cone* in $\mathfrak{a}$ are respectively given by

$$\begin{aligned} c_{\min} &= \sum_{\alpha \in \Delta_1} \mathbb{R}^+ L(h_\alpha) \subset \mathfrak{a}, \\ c_{\max} &= c_{\min}^* = \{H \in \mathfrak{a} : \alpha(H) \geq 0, \quad \forall \alpha \in \Delta_1\}. \end{aligned}$$

We check that the causal element satisfy $L(e') \in {c_{\min}}^\circ \subset {c_{\max}}^\circ$.

The Killing form on $\mathfrak{g}$ is denoted by B. The cone

$$C_{\max} = \{X \in \mathfrak{q} : \forall Y \in C_{\min} : B(X, Y) \geq 0\} = \overline{\mathrm{Ad}(H){c_{\max}}^\circ}.$$

is closed, pointed, generating and H-invariant in $\mathfrak{q}$ (see [8], chapter 2, for details). The semi-group associated is $S = \exp(C_{\max})H \subset G$. It is closed and satisfies the property : $S^\circ = H \exp {C_{\max}}^\circ = H \exp({c_{\max}}^\circ)H$.

The submanifold NAH is open in G and contains S°. Let us define $\overline{N}_D$ by the relation $\overline{N} \cap NAH = \overline{N}_0 \overline{N}_D$.

The domain $\overline{N}_D$ admits a description in terms of Jordan algebra. Consider the mapping :

$$n_{-1} : V^{pq} \to N_{-1} \quad z \mapsto \mathfrak{F}(e_p, z),$$

and the domain D of V^{pq} :

$$D = \{z \in V^{pq} \mid e_p - Q_p(z) \in \Omega_p\}, \tag{4}$$

with the notation $Q_p(z) = 2c_p(z^2) \in \overline{\Omega}_p$. Using the characterisation of the orbit $NAH \cdot e' = NA \cdot e' \subset V$ in terms of the Δ_i, we easily get the identification :

$$n_{-1}(D) = \overline{N}_D$$

Notice also that the linear isomorphism

$$\nu : V^{pq} \to \mathfrak{n}_1 \quad z \mapsto \{e_p, z, \cdot\} \tag{5}$$

identifies V^{pq} and $\mathfrak{n}_1$, and satisfies $\exp \circ \nu = n_1$. In the following, we will use the same notation to designate an object related to V^{pq} and its image by ν. ($D = \nu(D)$, for example.)

4. SPHERICAL FUNCTIONS ON G/H

The various objects used in this section are described in [5] : their definitions and properties are valid in the frame of ordered symmetric spaces, more general than the satellite cones.

A spherical function φ is a function defined on the interior S^o of the semi-group S, such that for all $x, y \in S^o$,

$$\int_H \varphi(xhy)dh = \varphi(x)\varphi(y),$$

where dh is a fixed Haar measure on H.

The Poisson kernel evaluated at $x \in NAH$ is the uniquely determined A-composant of x. Observe also that S is stable under the action of H :

$$h \cdot x \in S \subset NAH \quad \forall h \in H, \forall x \in S.$$

Consider a linear form $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the dual of $\mathfrak{a}_{\mathbb{C}}$ and $\rho \in \mathfrak{a}_{\mathbb{C}}^*$, the half-sum of the positive restricted roots, counted with multiplicities. Set also :

$$a_H(x)^\lambda = \exp\left(\lambda(\log a_H(x))\right).$$

For $x \in S^o$, we introduce :

$$\varphi_\lambda(x) = \int_H a_H(hx)^{\rho-\lambda} dh,$$

whenever the integeral is convergent. The domain of convergence of φ_λ is denoted by

$$\mathcal{E} = \left\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \forall x \in S^o, h \mapsto a_H(hx)^{\rho-\lambda} \in L^1(S^o)\right\}.$$

Finally, let us precise the definition of the Harish-Chandra function c_D associated with the ordered symmetric space G/H :

$$c_D(\lambda) = \int_{\overline{N}_D} a_H(\overline{n})^{\rho+\lambda} d\overline{n}.$$

The algebra of G-invariant differential operators on G/H is denoted by $\mathbb{D}(G/H)$. Let us summarize the main properties of the function φ_λ :

THEOREM 1. *The function φ_λ satisfy :*

- $\exists \chi_\lambda \in$*Hom*$(\mathbb{D}(G/H), \mathbb{C}) \mid \forall D \in \mathbb{D}(G/H)$, $D\varphi_\lambda = \chi_\lambda(D)\varphi_\lambda$.
- φ_λ *is a spherical function for* $\lambda \in \mathcal{E}$.
- $\varphi_\lambda = \varphi_{w\cdot\lambda}$ *for* $w \in \mathcal{W}$, *the Weyl group of* Δ.
- $\mathcal{E} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid |c_D(\lambda)| < \infty\}$.
- $\lim_{a\to+\infty} a^{\lambda-\rho}\varphi_\lambda(a) = c_0(\lambda)c_D(\lambda)$ *for Re*$(\lambda) \in \mathfrak{a}_+^* \cap \mathcal{E}$.

The first point is mentioned, for example, in lemma 8.2.4 of [8]. The four last points are coming from [5], respectively theorem 5.2, corollary 5.6, theorem 6.3, and theorem 6.8. In the last point, the notation $a \to +\infty$ means $\log(a) \in \mathfrak{a}^+$ and $\lim a^\alpha = 0$ for all restricted positive root $\alpha \in \Delta^+$, and the symbol c_0 refers to the Harish-Chandra c-function associated to the Riemannian symetric space $N_0A(H \cap K)/(H \cap K)$.

We also define the covector H_λ for $\lambda \in \mathfrak{a}^*$. First, define H'_λ such that for all $H \in \mathfrak{a}$, $B(H, H'_\lambda) = \lambda(H)$, and define after $H_\lambda = 2H'_\lambda/B(H'_\lambda, H'_\lambda)$. Hence, $\lambda(H_\lambda) = 2$ and in the case of satellite cones, $H_{\alpha_{ij}} = L(h_{ij})$.

The values of λ such that $|c_D(\lambda)| < \infty$, and an explicit formula of the function c_D where first determined for the satellite cones in [7], and after, for general ordered symmetric spaces in theorem III.5 of [10],

Theorem 2. *The set of λ such that the integral defining c_D converges is given by :*

$$\mathcal{E} = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \forall \alpha \in \Delta_1, \quad Re(\lambda(H_\alpha)) < 2 - m_\alpha\} \tag{6}$$

where m_α is the multiplicity of the root α and $H_\alpha \in \mathfrak{a}$ the covector associated to α. For $\lambda \in \mathcal{E}$, we also have :

$$c_D(\lambda) = \kappa \prod_{\alpha \in \Delta_1} B\left(\frac{m_\alpha}{2}, -\frac{1}{2}\lambda(H_\alpha) - \frac{m_\alpha}{2} + 1\right),$$

with B the Eulerian Beta function, and κ a positive constant, depending only on the pair $(\mathfrak{g}, \tau)$.

The previous Lie objects, in particular the Poisson kernel, can be described in terms of Jordan algebra. We have already explicited the domain D, in (4).

In the case of satellite cones, the set of the $\vec{\lambda} \in \mathbb{C}^r$ such that $|c(\lambda)| < \infty$ becomes :

$$\vec{\mathcal{E}} = \left\{\vec{\lambda} \in \mathbb{C}^r \mid \mathrm{Re}(\vec{\lambda}_i - \vec{\lambda}_{j+1}) > \frac{d}{2} - 1 \quad : 1 \leq i \leq p \leq j \leq r-1\right\} \tag{7}$$

The vector $\vec{\rho} \in \mathbb{C}^r$ corresponding to ρ is easiliy computed,

$$\vec{\rho}_i = \frac{d}{4}(2i - r - 1) \quad \forall i \in \{1, ..., r\}.$$

Before introducing the description of the Poisson kernel, we need the definition of the power function in a Jordan algebra.

Let $\mathbf{s} = (s_1, ..., s_r) \in \mathbb{C}^r$. For all $x \in V$ such that $\Delta_1(x)...\Delta_r(x) \neq 0$, we define :

$$\Delta_{\mathbf{s}}(x) = |\Delta_1(x)|^{s_1 - s_2} ... |\Delta_{r-1}(x)|^{s_{r-1} - s_r} |\Delta_r(x)|^{s_r},$$

Observe that this power function is slightly different of the one in [6], because of the absolute values appearing in the defintion. The main properties of the power functions are :

Proposition 1. *Fix $\mathbf{s} \in \mathbb{C}^r$. The power function $x \mapsto \Delta_{\mathbf{s}}(x)$ satisfy the following properties :*

- *Action of the minimal parabolic MAN : $\forall (m, a, n) \in M \times A \times N$,*

$$\Delta_{\mathbf{s}}(man \cdot x) = \chi_{\mathbf{s}}(a) \Delta_{\mathbf{s}}(x) \tag{8}$$

where $\chi_{\mathbf{s}}$ is the character defined on A by :

$$\chi_{\mathbf{s}}\left(P\left(\sum_{i=1}^{r} a_i c_i\right)\right) := a_1^{2s_1} ... a_r^{2s_r}, \quad a_i > 0 \quad \forall i \in \{1, ..., r\}.$$

- *Action of the group G : $\forall (g, x) \in G \times V$,*

$$\Delta(g \cdot x) = Det(g)^{\frac{r}{n}} \Delta(x). \tag{9}$$

These properties lead to :

Proposition 2. *Let $g \in NAH$. We have :*

$$a_H(g) = P\left(\sum_{i=1}^{r} \left| \frac{\Delta_i(ge')}{\Delta_{i-1}(ge')} \right|^{\frac{1}{2}} c_i\right),$$

with the convention $\Delta_0 = 1$. If $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, then $\vec{\mu} \in \mathbb{C}^r$ and :

$$a_H^{\mu}(g) = \Delta_{\vec{\mu}}(ge').$$

Proof. Let $g = nah$ with $n \in N$, $h \in H$ and $a = P(\sum_{i=1}^{r} a_i c_i)$ with $a_i > 0$, $\forall i \in \{1, ..., r\}$. From (8), we have :

$$\Delta_i(ge') = a_1^2 ... a_i^2 \Delta_i(e'),$$

and the first part of the proposition follows. We also observe :

$$P\left(\sum_{i=1}^{r} a_i c_i\right) = \exp\left(2 \sum_{i=1}^{r} \log(a_i) L(c_i)\right),$$

in consequence,

$$\begin{aligned}
a_H^{\mu}(g) &= \exp\left(\mu(\log a_H(g))\right) \\
&= \exp\left(\mu\left(\sum_{i=1}^{r} \log\left|\frac{\Delta_i(g \cdot e')}{\Delta_{i-1}(g \cdot e')}\right| L(c_i)\right)\right) \\
&= \exp\left(\sum_{i=1}^{r} \vec{\mu}_i \log\left|\frac{\Delta_i(g \cdot e')}{\Delta_{i-1}(g \cdot e')}\right|\right) \\
&= \Delta_{\vec{\mu}}(ge'),
\end{aligned}$$

which is the second claim. ∎

We end this section with the definition of a familiy of functions associated to the cone Ω', for $\tilde{\lambda} \in \mathbb{C}^r$:

$$\varphi_{\tilde{\lambda}} : S^\circ \cdot e' \to \mathbb{C}, \quad x \mapsto \int_H \Delta_{\tilde{\rho}-\tilde{\lambda}}(hx)dh.$$

We identify S^o/H with the futur $S^o \cdot e'$ of the base point in Ω'. Under this identification, this functions are indeed the spherical functions of the ordered symmetric space Ω' : $\varphi_{\tilde{\lambda}}(g \cdot e') = \varphi_\lambda(g)$.

5. ÓLAFSSON - PASQUALE CONJECTURE

There are multiple references to [14] in this section. Notice that in [14] the Lie objects are defined with respect to HAN instead of NAH. As in [5], we prefer NAH because this convention is more convenient to realize Ω' in the Jordan algebra V. It explains the minor differences between the formulae of [14] and the formulae proposed in this article.

Now, we focuson the meromorphic continuation of the spherical functions φ_λ, with respect to the parameter λ : the aim of the remaining sections is the determination of a small set containing the poles. This problem was also solved in [14], using an inexplicit method involving the Heckman-Opdam theory ([14], corollary 8.2) :

THEOREM 3. *For all $a \in S^o \cap A$, the function $\lambda \mapsto \varphi_\lambda(a)$ admits a meromorphic continuation to $\mathfrak{a}^*_{\mathbb{C}}$, with simple poles belonging to the polar set of the numerator of c_D.*

In the same article, G. Ólafsson and A. Pasquale suggest another approach to this problem, more explicit, using Bernstein-Sato identities. We will develop this tool.

We have (formula (9) page 359, [14]) for every $a \in S^o \cap A$ and $\lambda \in \mathcal{E}$:

$$\varphi_\lambda(a) = \int_{\overline{N}_D} \left[\int_{K\cap H} a_H(\omega k a)^{\rho-\lambda} dk \right] a_H(\omega)^{\lambda+\rho} d\omega, \tag{10}$$

where the integrated function between brackets is regular, with support far from the singularties.

The formula (10) shows that the problem of meromorphic extension of spherical functions can be solved by the construction of Bernstein-Sato identities related to the Poisson kernel a_H. Following [14], we will consider the interpretation of $a_H^{\lambda+\rho}$ as a product of polynomials on $\mathfrak{n}_{-1}$, raised to different complex powers.

Recall the definition (2) of the fundamental weight μ_i associated to a simple root α_i. In the context of ordered symmetric spaces, $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ are conjugate. Then the K-spherical representations are also H-sphercial representations. For $j = 1, ..., r$, we designate by π_j the representation of highest weight μ_j. The representation space can be endowed with a scalar product satisfying $\pi_j(.)^* = \pi_j(\theta(.)^{-1})$. Let v_j and u_j be respectively the highest weight and a H-fix vector such that $(v_j, u_j) = 1$. Finally, we identify $\mathfrak{n}_{-1}$ with N_{-1} via : $\exp : \mathfrak{n}_{-1} \to N_{-1}, \quad \mathbf{t} \mapsto \exp(\mathbf{t})$.

By definition, a function f on N_{-1} is a polynomial if $f(\exp(\mathbf{t}))$ is a polynomial function on $\mathfrak{n}_{-1}$. The elements of N_{-1} are nilpotent, hence the matrix coefficients of π_j are polynomials on N_{-1}. Thus define :

$$p_j(\mathbf{t}) = |(\pi_j(\exp(\mathbf{t}))^{-1} v_j, u_j)|^2, \quad \forall j \in \{1, ..., r\}.$$

Again, this result is slightly different from [14], because of our conventions. Consider the decomposition with respect to NAH of $\exp(\mathbf{t})$:

$$\exp(\mathbf{t}) = n(\exp(\mathbf{t})) a_H(\exp(\mathbf{t})) h(\exp(\mathbf{t})).$$

Then,

$$\begin{aligned} p_j(\mathbf{t}) &= |\pi_j^*(h(\exp(\mathbf{t}))^{-1})\pi_j(a_H(\exp(\mathbf{t}))^{-1})\pi_j(n(\exp(\mathbf{t}))^{-1})v_j, u_j)|^2 \\ &= |(\pi_j(a_H(\exp(\mathbf{t}))^{-1})v_j, \pi_j(\theta(h(\exp(\mathbf{t}))))u_j)|^2 \\ &= a_H(\exp(\mathbf{t}))^{-\mu_j}. \end{aligned}$$

Therefore :

$$a_H(\omega_{\mathbf{t}})^{\lambda+\rho} = \prod_{j=1}^{r} p_j(\mathbf{t})^{z_j(\lambda+\rho)},$$

with

$$z_j(\lambda+\rho) = \frac{1}{4}\,(\lambda+\rho)\,(H_\alpha),$$

and the p_j are non-negative polynomials on $\log(\overline{N}_D)$.

Before introducing Bernstein identities, let us complete the notations. Let E be a vector space. We denote by $\mathbb{K}[E]$ (resp. $\mathbb{K}\{E\}$) the algebra of polynomials (resp. rational functions) on E with coefficients in $\mathbb{K}$. We also write $\mathbb{C}[\lambda]$ instead of $\mathbb{C}[\lambda_1, ..., \lambda_r]$.

Assume that we have the Bernstein identitiy :

$$Q(\lambda, \mathbf{t}, \partial) a_H(\omega_{\mathbf{t}})^{\lambda+\rho} = b(\lambda) a_H(\omega_{\mathbf{t}})^{\lambda+\rho-\delta}, \tag{11}$$

with $b \in \mathbb{C}[\mathfrak{a}_{\mathbb{C}}^*]$, $Q \in \mathbb{C}[\mathfrak{a}_{\mathbb{C}}^*, \mathfrak{n}_{-1}, \mathfrak{n}_{-1}]$, and $\delta \in \mathfrak{a}^*$ a suitable shift on the exponent. Using this identity and formula (10), we obtain a meromorphic continuation of φ_λ, and a localistion of its poles. The authors of [14] formulate the following conjecture, page 375 :

Conjecture 1. The formula (11) holds with the following shift δ and Bernstein polynomial b :

$$\begin{aligned} \delta &: \quad (\delta, \alpha_j) = 2(\alpha_j, \alpha_j), \quad \forall \alpha_j \in \Pi. \\ b(\lambda) &= \prod_{\alpha \in \Delta_1} \left(\frac{1}{2}\lambda(H_\alpha) + \frac{m_\alpha}{2} - \frac{1}{2}\delta(H_\alpha) + 1\right) ... \left(\frac{1}{2}\lambda(H_\alpha) + \frac{m_\alpha}{2}\right). \end{aligned}$$

We show that the conjecture 1 does not hold in all situations :

PROPOSITION 3. *We consider a vector space E, and $f_1, ..., f_r \in \mathbb{C}[E]$, non-negative polynomials on an open set $\Omega \subset E$. Fix $k_1, ..., k_r \in \mathbb{N}$, with $k_1 > 0$. Assume that for all $\alpha = (\alpha_1, ..., \alpha_r) \in \mathbb{C}$ and $\mathbf{z} \in \Omega$ we have :*

$$b(\alpha_1, ..., \alpha_r) f_1^{\alpha_1 - k_1}(\mathbf{z}) ... f_r^{\alpha_r - k_r}(\mathbf{z}) = Q(\alpha, \mathbf{z}, \partial) f_1^{\alpha_1}(\mathbf{z}) ... f_r^{\alpha_r}(\mathbf{z}), \qquad (12)$$

where $b \in \mathbb{C}[\alpha]$ is a product of affine factors, $Q \in \mathbb{C}[\alpha, E, E]$.

Furthermore, assume that there exists $\mathbf{z_0}$ in the boundary of Ω such that $f_1(\mathbf{z_0}) = 0$, and $f_2(\mathbf{z_0}) ... f_r(\mathbf{z_0}) \neq 0$.

Then, the monomial α_1 divides b.

Proof. If α_1 divides Q, the result is trivial. We assume that α_1 is not a divisor of Q. Choose $\alpha_i \in \mathbb{N}$ with $\alpha_i \geq k_i$ for $i = 2, ..., r$, and evaluate (12) at $\alpha_1 = 0$:

$$b(0, \alpha_2, ..., \alpha_r) f_1^{-k_1}(\mathbf{z}) f_2^{\alpha_2 - k_2}(\mathbf{z}) ... f_r^{\alpha_r - k_r}(\mathbf{z}) = Q(\alpha, \mathbf{z}, \partial) f_2^{\alpha_2}(\mathbf{z}) ... f_r^{\alpha_r}(\mathbf{z})$$

the right hand-side is bounded on a neighborhood of the point $\mathbf{z_0}$, though the left hande-side is not, except if

$$b(0, \alpha_2, ..., \alpha_r) = 0. \qquad (13)$$

In consequence, (13) holds for all integers α_i such that $\alpha_i \geq k_i$, and $i = 2, ..., r$. We deduce α_1 divides b. ∎

LEMMA 2. *There are ordered symmetric spaces G/H satisfying the following properties : there exist a compact simple root α_j and $z \in \mathfrak{n}_1$ such that $p_i(z) \neq 0$ if $i \neq j$, and $p_j(z) = 0$.*

Proof. We consider a satellite cone of type Ω', with $q \geq 2$ and $p = 1$. In this particular case, an explicit computation using formula (1) leads to :

$$\Delta_j(n_{-1}(z)e') = 1 - \sum_{i=j}^{r-1} ||z_i||^2.$$

Choose z_1 and z_2 unitary, and $z_3 = ... = z_{r-1} = 0$. Then $|\Delta_j(n_{-1}(z)e')| \neq 0$ for every $j \neq 2$ and $|\Delta_{r-1}(n_{-1}(z)e')| = 0$, but α_2 is a compact simple root.

A consequence of $|\Delta(n_{-1}(z)e')| = 1$, is that the level line $|\Delta| = 1$ in Ω' is a semi-simple ordered symmetric space with the expected property. ∎

COROLLARY 1. *The conjecture 1 must be rectified.*

Proof. We consider an ordered symmetric space G/H as in lemma 2. Using the last proposition on $p_1, ..., p_r$, we find :

$$2z_i(\lambda + \rho) = \frac{1}{2}(\lambda + \rho)(H_{\alpha_i}) = \frac{1}{2}\lambda(H_{\alpha_i}) + \frac{m_{\alpha_i}}{2} \text{ divides } b(\lambda).$$

but the only factor of the conjecture's polynomial b associated to a simple root corresponds to the non-compact one. This is a contradiction. ∎

The shape of the domain $\mathcal{E}$, formula (6), indicates that the shift $\delta = \mu_p$ in the direction of the non-compact simple root α_p is sufficient to get the meromorphic continuation on $\mathfrak{a}^*_{\mathbb{C}} : \mathcal{E} - \mathbb{N}\delta = \mathbb{C}^r$. This remark suggest to state a slightly modified version of conjecture 1, with a shift only in the direction of the non-compact simple root α_p :

Conjecture 2. The formula (11) holds with the following shift δ and Bernstein polynomial b :

$$\begin{aligned} \delta &= \mu_p \\ b(\lambda) &= \prod_{\alpha\in\Delta_1} \left(\frac{1}{2}\lambda(H_\alpha) + \frac{m_\alpha}{2} - \frac{1}{2}\delta(H_\alpha) + 1\right) \dots \left(\frac{1}{2}\lambda(H_\alpha) + \frac{m_\alpha}{2}\right). \end{aligned}$$

We remark that for rank one ordered symmetric spaces, the conjectures 1 and 2 are the same. The conjecture 1 as been proved in [14], section 6, in the rank one case. In the following we will prove a slightly weaker version of conjecture 2, in the context of satellite cones. The result will allow us to determine the poles of spherical functions :

Conjecture 3. The greatest common divisor of the polynomials satisfying formula (11) with the shift $\delta = \mu_p$, is :

$$b_{\mathrm{gcd}}(\lambda) = \prod_{\alpha\in\Delta_1} \left(\frac{1}{2}\lambda(H_\alpha) + \frac{m_\alpha}{2} - \frac{1}{2}\delta(H_\alpha) + 1\right) \dots \left(\frac{1}{2}\lambda(H_\alpha) + \frac{m_\alpha}{2}\right).$$

6. PROOF OF THE CONJECTURE ON SATELLITE CONES

The proof of conjecture 3 is based on theorem 5.2 in [1], where the author exhibates Bernstein-Sato identities for the power functions of an Euclidean Jordan algebra. The greatest common divisor of the Bernstein polynomials appearing in these identities is also computed :

Theorem 4. *There exist $B \in \mathbb{C}[\mathbf{s}]$ and $E \in \mathbb{C}[\mathbf{s}, V, V]$ such that*

$$E(\mathbf{s}, x, \partial)\Delta_{\mathbf{s}}(x) = B(\mathbf{s})\Delta_{\mathbf{s}}(x)\Delta_p(x)^{-1}. \tag{14}$$

Furthermore, the greatest common divisor of the polynomials B satisfying the Bernstein identity (14) is

$$B_{\mathrm{gcd}}(\mathbf{s}) = \prod_{1\le i\le p\le j\le r} \left(s_i - s_{j+1} + \frac{d}{2}(j-i)\right),$$

with the convention $s_{r+1} = 0$.

In order to prove conjecture 3, the strategy is the use of the radial part of differential operators appearing in theorem 4, with respect to a suitable coordinate system. We begin with the description of the convex cone Ω in terms of Iwasawa coordinates. Let u be an element of V. We consider the decompostion of u in the fixed Jordan frame :

$$u = \sum_{i=1}^{r} u_i c_i + \sum_{1 \leq i < j \leq r} u_{ij}.$$

We set $V^+ = \{u \in V : u_i > 0, i = 1, ..., r\}$. As in [6] page 112, we define :

$$\begin{aligned} a_i(u) &: = P(c_1 + ... + c_{i-1} + u_i c_i + c_{i+1} + ... + c_r) \quad \forall i \in \{1, ..., r\}, \\ n_i(u) &: = \mathfrak{F}(c_i, u_{ii+1} + ... + u_{ir}) \quad \forall i \in \{1, ..., r-1\}, \\ t(u) &: = a_1(u) n_1(u) a_2(u) n_2(u) ... n_{r-1}(u) a_r(u). \end{aligned}$$

The results VI.3.8 and VI.3.9 of [6] are summarized in the following proposition :

Proposition 4. *The mapping $T : u \mapsto t(u)e$ is a diffeomorphism from V^+ onto Ω'. Its Jacobian is*

$$\det(D_u T_V) = 2^r \left(\prod_{k=1}^{r} u_k^{d(r-k)+1} \right).$$

We denote by t_p (resp. t_q^*) the mapping t related to V_p (resp. V_q^*) instead of V. Observe that the mapping $t_p(u)$ and $t_q^*(v)$ extend to V. Now, consider the parametrization of a neighborhood of e' in V :

Proposition 5. *With the notations of proposition 4, we define :*

$$\begin{aligned} \Psi &: \quad V_q^+ \times (V_p^*)^+ \times D \to \Omega', \\ (u, v, z) &\mapsto \quad t_q^*(v) t_p(u) n_{-1}(z) e', \end{aligned}$$

The mapping Ψ is a diffeomorphism onto an open set of Ω'. The absolute value of its Jacobian is :

$$|\det(D_{(u,v,z)}\Psi)| = 2^r \prod_{k=1}^{p} u_k^{d(r-k)+1} \prod_{l=1}^{q} v_l^{d(r-l)+1}.$$

Proof. We denote by x_γ the composant of $x \in V$ with respect to $V(e_p, \gamma)$ in the Peirce decomposition associated to the idempotent e_p. Define $x = t_q^*(v) t_p(u) n_{-1}(z) e'$, and denote by h the map from $V_p^+ \times V^{pq}$ to V^{pq} which associate to (u, z) the element $t_p(u) z$. Formula (1) leads to :

$$\begin{cases} x_1 = t_p(u)\{e_p - Q_p(z)\} \\ x_{1/2} = -t_q^*(v) t_p(u) z \\ x_0 = -t_q^*(v) e_q^* \end{cases},$$

hence,

$$\left\{\begin{array}{l} x_1 = t_p(u)e_p - Q_p(h(u,z)) \\ x_{1/2} = -t_q^*(v)t_p(u)z \\ x_0 = -t_q^*(v)e_q^* \end{array}\right. .$$

Now, let us denote by $J = \det(D_{(u,v,z)}\Psi)$, the Jacobian studied. Then,

$$\begin{aligned}
|J| &= \left|\det\begin{pmatrix} D_u x_1 & D_v x_1 & D_z x_1 \\ D_u x_{1/2} & D_v x_{1/2} & D_z x_{1/2} \\ D_u x_0 & D_v x_0 & D_z x_0 \end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} D_u\left\{t_p(u)e_p - Q_p(h(\cdot,z))\right\} \\ -t_q^*(v)D_u h(\cdot,z) \\ 0 \end{pmatrix}\right. \\
&\qquad \left.\begin{matrix} 0 & -D_z\left\{Q_p(h(u,\cdot))\right\} \\ D_v x_{1/2} & -t_q^*(v)t_p(u)D_z(z) \\ -D_v\{t_q^*(v)e_q^*\} & 0 \end{matrix}\right)\Bigg| \\
&= \left|\det\begin{pmatrix} D_u\{t_p(u)e_p\} - D_u\left\{Q_p\circ h(\cdot,z)\right\} & D_z Q_p\circ h(u,\cdot) \\ t_q^*(v)D_u h(\cdot,z) & -t_q^*(v)t_p(u) \end{pmatrix}\right. \\
&\qquad \cdot \det\left(D_v\left\{t_q^*(v)e_q^*\right\}\right)\Big|
\end{aligned}$$

We decompose the M matrix appearing in the first factor of the right hand-side into a product of two triangular block matrices (of determinant 1) and one diagonal block matrix :

$$\begin{aligned}
M &= \begin{pmatrix} 1 & \left\{D_{h(u,z)}Q_p\right\}t_q^*(v)^{-1} \\ 0 & 1 \end{pmatrix} \\
&\qquad \begin{pmatrix} D_u\{t_p(u)e_p\} & 0 \\ 0 & -t_q^*(v)t_p(u) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t_p(u)^{-1}D_u h(\cdot,z) & 1 \end{pmatrix}
\end{aligned}$$

since, using basic differential calculus, we have :

$$\begin{aligned}
M_{1,2} &= -D_z\left\{Q_p\circ h(u,\cdot)\right\} \\
&= -\left\{D_{h(u,z)}Q_p\right\}\circ t_p(u) \\
&= \left\{D_{h(u,z)}Q_p\right\}t_q^*(v)^{-1}\circ\left(-t_q^*(v)t_p(u)\right) \\
M_{2,1} &= -t_q^*(v)D_u h(\cdot,z) \\
&= \left(-t_q^*(v)t_p(u)\right)\circ\left(t_p(u)^{-1}\right)D_u h(\cdot,z) \\
M_{1,1} &= D_u\{t_p(u)e_p\} - D_{h(u,z)}Q_p\circ D_u h(\cdot,z) \\
&= D_u\{t_p(u)e_p\} - D_{h(u,z)}Q_p\circ D_u h(\cdot,z) \\
&= D_u\{t_p(u)e_p\} + \{D_{h(u,z)}Q_p\}t_q^*(v)^{-1}\circ\left(-t_q^*(v)t_p(u)\right) \\
&\quad \circ\left(t_p(u)^{-1}D_u h(\cdot,z)\right)
\end{aligned}$$

Inserting the result of proposition 4, we get :

$$\begin{aligned}
|J| &= \left|\det(M)\det(D_v\{t_q^*(v)e_q^*\})\right| \\
&= \left|\det\begin{pmatrix} D_u\{t_p(u)e_p\} & 0 \\ 0 & -t_p(u)t_q^*(v)\end{pmatrix}\det(D_v\{t_q^*(v)e_q^*\})\right| \\
&= \left|\det(D_u\{t_p(u)e_p\})\det(-t_p(u)t_q^*(v)|_{V^{pq}})\det(D_v\{t_q^*(v)e_q^*\})\right| \\
&= \left|\det(D_u\{t_p(u)e_p\})\det(D_v\{t_q^*(v)e_q^*\})\right| \prod_{j=1}^{p} u_j^{dq}\prod_{j=1}^{q} v_j^{dp} \\
&= 2^r\prod_{k=1}^{p} u_k^{d(r-k)+1}\prod_{l=1}^{q} v_l^{d(r-l)+1}
\end{aligned}$$

which is the expected result. ∎

The fact that the last Jacobian is independant of z has an important consequence :

COROLLARY 2. *Let $E \in \mathbb{C}[V,V]$. Via the diffeomorphism Ψ, it corresponds to E a differential operator $E'(u,v,z,\partial')$ on $V_q^+ \times (V_p^*)^+ \times D$, defined by the relation :*

$$E'(u,v,z,\partial')\{f\circ\Psi\} = (E(x,\partial)f)\circ\Psi.$$

Then $E' \in \mathbb{C}\{V_q^+,(V_p^*)^+\}[V^{pq},V^{pq}]$: in particular, $E'(u,v,z,\partial)$ is a polynomial with respect to z.

Proof. We denote by $J(u,v,z)$ the Jacobian matrix of Ψ. The transpose of the cofactors matrix of $J(u,v,z)$ is $\widetilde{J}(u,v,z)$. We have

$$x = \Psi(u,v,z) \in \mathbb{R}[V_p,V_q^*,V^{pq}], \tag{15}$$

so $J(u,v,z) \in \mathbb{R}[V_p,V_q^*,V^{pq}]$ and $\widetilde{J}(u,v,z) \in \mathbb{C}[V_p,V_q^*,V^{pq}]$.

Since $\det(J(u,v,z))$ depends only on u and v,

$$J(u,v,z)^{-1} = \widetilde{J}(u,v,z)/\det(J(u,v,z)) \in \mathbb{R}\{V_p,V_q^*\}[V^{pq}]. \tag{16}$$

From equations (15) and (16), we deduce that a differential operator on V with polynomial coefficients is, under the change of variable Ψ, a differential operator on $V_q^+ \times (V_p^*)^+ \times D$, polynomial with respect to z and rational with respect to u,v. ∎

The following statement generalizes the theorem 3.6, page 259, in [9], which shows the existence of the radial part of differential operators acting on smooth invariant functions. The theorem extends to the context of relatively invariant functions :

THEOREM 5. *Let V be a smooth, real, second countable manifold. assume that a Lie group H acts on V, and satisfy the following axiom of transversality for a submanifold W of V :*

$$T_w V = (H \cdot w)_w \oplus T_w W, \quad \forall w \in W.$$

Let χ be a character of H. Let E be a differential operator on V. Then there exists a unique differential operator $\dagger_\chi(E)$ on W such that :

$$(Ef)|_W = \dagger_\chi(E)\left\{ f|_W \right\},$$

on functions f, relatively invariant with respect to χ ($f(hx) = \chi(h)f(x)$), and smooth on an open set of V.

Proof. The proof is the same as in [9], except that we build a relatively invariant function of $C^\infty(V_0)$, starting from a function $\phi \in C^\infty(W_0)$, by the formula :

$$f(h \cdot x) = \chi(h)\phi(x), \forall x \in W_0, \forall h \in H,$$

where $W_0 \subset W$ is an open set and V_0 is the open set $H \cdot W_0$. ∎

THEOREM 6. *There exist polynomials $Q \in \mathbb{C}[\tilde{\mathbf{s}}, V, V]$, and $b \in \mathbb{C}[\tilde{\mathbf{s}}]$, such that for all $z \in D \subset V^{pq}$,*

$$Q(\tilde{\mathbf{s}}, z, \partial)\Delta_{\tilde{\mathbf{s}}}(n_{-1}(z)e') = b(\tilde{\mathbf{s}})\Delta_{\tilde{\mathbf{s}}}(n_{-1}(z)e')\Delta_p(n_{-1}(z)e')^{-1}. \tag{17}$$

Moreover the greatest common divisor of the polynomials b which satisfy the relation (17) is :

$$b_{\mathrm{gcd}}(\tilde{\mathbf{s}}) = \prod_{1 \le i \le p \le j \le r-1} \left(\vec{s}_i - \vec{s}_{j+1} + (j-i)\frac{d}{2} \right).$$

Proof. Consider the character $\chi_{\tilde{\mathbf{s}}}$ defined on $g \in N_p A_p \times N_q^* A_q^*$, by :

$$\chi_{\tilde{\mathbf{s}}}(g)\Delta_{\tilde{\mathbf{s}}}(x) = \Delta_{\tilde{\mathbf{s}}}(gx).$$

According to theorem 4, let $E \in \mathbb{C}[\tilde{\mathbf{s}}, V, V]$ and $b \in \mathbb{C}[\tilde{\mathbf{s}}]$ satisfying :

$$E(\tilde{\mathbf{s}}, x, \partial)\Delta_{\tilde{\mathbf{s}}}(x) = b(\tilde{\mathbf{s}})\Delta_{\tilde{\mathbf{s}}}(x)\Delta_p(x)^{-1}.$$

First, we remark that the radial part $\dagger_{\chi_{\tilde{\mathbf{s}}}}(E(\tilde{\mathbf{s}}, x, \partial))$ is again polynomial in $\tilde{\mathbf{s}}$: we write the operator $E(\tilde{\mathbf{s}}, x, \partial)$ in terms of coordinates (u, v, z), and apply it on a function of the shape : $\chi_{\tilde{\mathbf{s}}}(t_p(u)t_q^*(v))f(n_{-1}(z)e')$. After the evaluation at

$$t_p(u)t_q^*(v) = 1,$$

the characters take the value 1, and the result is a combination of polynomials in $\tilde{\mathbf{s}}$ and partial derivatives of $f(n_{-1}(z)e')$.

Theorems 6 and 5, imply that the radial part $\dagger_{\chi_{\tilde{\mathbf{s}}}}(E(\tilde{\mathbf{s}}, x, \partial))$ satisfies the equation :

$$\dagger_{\chi_{\tilde{\mathbf{s}}}}(E(\tilde{\mathbf{s}}, x, \partial))\Delta_{\tilde{\mathbf{s}}}(n_{-1}(z)e') = B(\tilde{\mathbf{s}})\Delta_{\tilde{\mathbf{s}}}(n_{-1}(z)e')\Delta_p(n_{-1}(z)e')^{-1}. \quad (18)$$

Hence, the greatest common divisor b_{gcd} the polynomials B of theorem 4, and therefore divides the polynomial B_{gcd}. Furthermore, the operator

$$E(\tilde{\mathbf{s}} + k(\imath_1 + ... + \imath_r), x) \circ \Delta_r(x)^k$$

satisfies also the relation (18), for all positive integer k. Thus,

$$b_{\mathrm{gcd}} \text{ divides } B_{\mathrm{gcd}}(\tilde{\mathbf{s}} + k(\imath_1 + ... + \imath_r)) \quad \forall k \in \mathbb{N}.$$

Computing the greatest common divisor of this family of polynomials, we get :

$$b_{\mathrm{gcd}}(\tilde{\mathbf{s}}) \text{ divides } \prod_{1 \leq i \leq p \leq j \leq r-1} \left(\vec{s}_i - \vec{s}_{j+1} + (j-i)\frac{d}{2}\right).$$

Conversely, the equality is given by the corollary of proposition 5.1 in [14]. The result is based on the function c_D, but the particular shape of the shift δ does not play any role, so the proposition still holds with our choice $\delta = \mu_p$. ∎

We have proved the conjecture 3 in the context of satellite cones. Notice that the conjecture 2 holds at least for some of them :

Example 1. Consider the particular case of the satellite cones Ω' with parameters $p = 2$ and $q = 1$. With notations of Proposition 3.6 in [1], equation (14) holds for the operator $E = D_{r-1}(\vec{s} - \vec{\imath}_r, x) \circ \Delta_r(\partial) - \Delta_r(\partial) \circ D_{r-1}(\vec{s}, x)$, and for the polynomials

$$(\vec{s}_2 - \vec{s}_3)(\vec{s}_1 - \vec{s}_2 + d/2) = B_{\mathrm{gcd}}(\tilde{\mathbf{s}})$$

Hence, the conjecture 2 is checked for the ordered symmetric spaces coming from the symmetric pairs :

$$(\mathfrak{sl}(3,\mathbb{R}), \mathfrak{so}(2,1)); (\mathfrak{sl}(3,\mathbb{C}), \mathfrak{su}(2,1)); (\mathfrak{su}^*(6), \mathfrak{sp}(2,1)); (\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)}).$$

Finally, we state the following result, which is an important application of the Bernstein identities on ordered symmetric spaces :

Theorem 7. *For all $a \in S^o \cap A \cdot e' \subset \Omega'$, the function $\vec{\lambda} \mapsto \varphi_{\vec{\lambda}}(a)$ admits a meromorphic continuation to $\mathbb{C}^r$, with simple poles belonging to the polar set of the numerator of the function c_D. More precisely :*

$$\vec{\lambda} \mapsto \prod_{1 \leq i \leq p \leq j \leq r-1} \frac{1}{\Gamma\left(\vec{\lambda}_i - \vec{\lambda}_{j+1} + 1\right)} \varphi_{\vec{\lambda}}(a)$$

admits an holomorphic continuation to $\mathbb{C}^r$.

Proof. From [14], for every compact subset $L(C)$ of $\mathfrak{a}$ and for every $\vec{\lambda} \in \mathbb{C}^r$, the following function is smooth :

$$\begin{array}{rcl} g_{\vec{\lambda}} : \overline{D} \times C & \to & \mathbb{C}, \\ (z,a) & \mapsto & \displaystyle\int_{K\cap H} \Delta_{\vec{\rho}-\vec{\lambda}}(n_{-1}(z)ka \cdot e')dk. \end{array}$$

Consider a polynomial b and a polynomial $Q \in \mathbb{C}[\vec{\lambda}, V^{pq}, V^{pq}]$, satisfying the Bernstein identity of theorem 6. According to formula (10), we have :

$$\begin{array}{rcl} \varphi_{\vec{\lambda}}(a) & = & \displaystyle\frac{b(\vec{\rho}+\vec{\lambda}+\vec{\delta})}{b(\vec{\rho}+\vec{\lambda}+\vec{\delta})} \int_D \Delta_{\vec{\rho}+\vec{\lambda}}(n_{-1}(z)e')g_{\vec{\lambda}}(z,a)dz \\ & = & \displaystyle\frac{1}{b(\vec{\rho}+\vec{\lambda}+\vec{\delta})} \int_D Q(\vec{\lambda}+\vec{\rho}+\vec{\delta}, z, \partial)\Delta_{\vec{\rho}+\vec{\lambda}+\vec{\delta}}(n_{-1}(z)e')g_{\vec{\lambda}}(z,a)dz \\ & = & \displaystyle\frac{1}{b(\vec{\rho}+\vec{\lambda}+\vec{\delta})} \int_D \Delta_{\vec{\rho}+\vec{\lambda}+\vec{\delta}}(n_{-1}(z)e')Q'(\vec{\lambda}+\vec{\rho}+\vec{\delta}, z, \partial)g_{\vec{\lambda}}(z,a)dz \end{array}$$

where Q' is the adjoint operator of Q with respect to the measure dz.

The integral defining the right hand-side converges for all $\vec{\lambda}$ such that

$$\lambda \in \vec{\mathcal{E}} - \vec{\delta}.$$

We check on fromula (7) that the iteration of this process leads to the meromorphic continuation on $\mathbb{C}^r$. Moreover, the poles are localized among the translated by $-\mathbb{N}$ of the zero set of the polynomial b. This process can be applied for all the Bernstein polynomials b, hence, by unicity of the holomorphic continuation, the poles are among the translated of the zero set of greatest common divisor :

$$b_{\mathrm{gcd}}(\vec{\lambda}+\vec{\rho}) = \prod_{1\leq i\leq p\leq j\leq r-1} \left(\vec{\lambda}_i - \vec{\lambda}_{j+1}\right),$$

which is the expected result. ∎

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